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On the eigenvalues of the sinc^2 kernel

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Abstract. The eigenvalues of the homogeneous Fredholm integral equation, whose kernel is a $\sin^2 x/x^2$ function, are shown to be non-degenerate. An upper bound for the largest eigenvalue is established and an approximate formula for evaluating the eigenvalues is suggested. Applications of these results to speckle and photocounting statistics are presented.

1. Introduction

Among homogeneous Fredholm integral equations of the second kind, those whose kernel can be identified with an autocorrelation function play a very important role. They are the starting points for techniques like the Karhunen–Loeve expansion and the Kac–Siegert analysis (eg Davenport and Root 1958). As such, they are widely used in the realm of information processing of both electronic (Helstrom 1968) and optical (Frieden 1972) kind. The case of the $\sin x/x$ kernel has been thoroughly studied and its eigenfunctions and eigenvalues are well known (Slepian and Pollack 1961). Indeed, a lot of applications have been carried out by the use of such eigenfunctions or of their two-dimensional generalizations (Slepian 1964). On the other hand, little information is available for the case of the $\sin^2x/x^2$ kernel, whose significance is particularly appreciated in optics (eg Goodman 1968). The corresponding integral equation, whose solutions are not analytically known, has been treated numerically or by perturbative methods so that numerical results and approximate solutions are available (Fedotowski 1972 and Bendinelli *et al* 1974). Furthermore, general upper and lower bounds for the eigenvalues have been found (Gori 1974).

In this paper, we will give some additional results about the solutions pertaining to the $\sin^2x/x^2$ kernel. First, we will prove that the eigenvalues are not degenerate. Secondly, we will establish an upper bound for the first eigenvalue (ie the eigenvalue of maximum modulus) that improves previously known bounds. This result suggests the introduction of an approximate formula for computing the eigenvalues. We will discuss such a formula and its validity. As an example, we will show the application of these results to speckle and photocounting statistics.

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Throughout the paper, the following notation will be used:

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| \leqslant \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases}$$

$$\Lambda(x) = \begin{cases} 1-|x| & |x| \leqslant 1 \\ 0 & |x| > 1 \end{cases} \tag{1}$$

$$\mathrm{sinc}(x) = \sin \pi x/\pi x.$$

The symmetrical form of the Fourier transform will be used. Unless otherwise stated, the Fourier transform will be denoted by a tilde:

$$\tilde{f}(\nu) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi \mathrm{i}\nu x}\, \mathrm{d}x. \tag{2}$$

Finally $[x]$ means the largest integer less than or equal to x.

2. Non-degeneracy of eigenvalues

Let us consider the Fredholm homogeneous integral equation

$$\mu_n \phi_n(x) = \int_{-a/2}^{a/2} \phi_n(y)\, \mathrm{sinc}^2(x-y)\, \mathrm{d}y. \tag{3}$$

For this equation a denumerably infinite set of eigenfunctions $\phi_n(x)$ corresponding to real and less than unity eigenvalues μ_n exists (Gori 1974). Both the eigenfunctions and the eigenvalues depend on the parameter a that we will term the space-bandwidth product†. We want to show that the eigenvalues are non-degenerate. This is relevant information when the eigenfunctions and the eigenvalues of equation (3) are used for physical applications. In a number of cases (eg image formation processes) the eigenfunctions are identified with the degrees of freedom of a physical system each of them being characterized by a weighting factor (eigenvalue). If an eigenvalue exhibits a degeneracy of order n, the number of degrees of freedom to be associated with it equals n. Therefore, the existence of degeneracy must be known when the number of degrees of freedom of a system is evaluated through the eigenvalues. As another example, in statistical applications (see § 4) the existence of degeneracy affects the structure of probability density functions. The demonstration of our thesis will be made for a general class of kernels, which includes $\mathrm{sinc}^2(x)$.

Indeed, let us consider the integral equation

$$\mu\phi(x) = \int_{-a/2}^{a/2} \phi(y) K(x-y)\, \mathrm{d}y \tag{4}$$

where the subscript n appearing in equation (3) has been dropped for the sake of simplicity, a is finite and $K(x)$ is the inverse Fourier transform (FT) of an even real function $p(\nu)$, defined in the interval $(-\nu_M, +\nu_M)$, with $p(\nu) \to 0$ when $\nu \to -\nu_M^+$ or $\nu \to +\nu_M^-$:

$$K(x) = \int_{-\nu_M}^{\nu_M} p(\nu)\, \mathrm{e}^{2\pi \mathrm{i}\nu x}\, \mathrm{d}\nu \tag{5}$$

† The reader should be cautioned that authors using the asymmetrical form of Fourier transform, refer to a parameter c related to a by $a = 2c/\pi$.

so that $K(x)$ is real and symmetric. ν_M may be either finite or infinite. We observe that $\mathrm{sinc}^2 x$ belongs to the class of kernels described by equation (5) with $p(\nu) = \Lambda(\nu)$ and $\nu_M = 1$.

Let us also consider the kernel $T(x)$

$$T(x) = \int_{-\nu_M}^{\nu_M} \nu p'(\nu)\, \mathrm{e}^{2\pi i \nu x}\, \mathrm{d}\nu. \tag{6}$$

We maintain that the eigenvalues μ of equation (4) are not degenerate when $T(x)$ is definite (see Pogorzelski 1966 for the definiteness condition of a symmetric kernel). Indeed let us establish the following.

Lemma
No eigenfunction of equation (4) can simultaneously vanish at $x = a/2$ and $x = -a/2$, when $T(x)$ is definite.

Proof
Differentiation of equation (4), multiplication of both sides by $x\phi(x)$ and integration in the interval $(-a/2, a/2)$ yields

$$\mu \int_{-a/2}^{a/2} x\phi(x)\phi'(x)\,\mathrm{d}x = \int_{-a/2}^{a/2}\int_{-a/2}^{a/2} x\phi(x)\phi(y)\frac{\mathrm{d}}{\mathrm{d}x}K(x-y)\,\mathrm{d}x\,\mathrm{d}y. \tag{7}$$

Exploiting the symmetry of $K(x)$, the right-hand side of equation (7) can be written as

$$\frac{1}{2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} (x-y)\phi(x)\phi(y)\frac{\mathrm{d}}{\mathrm{d}x}K(x-y)\,\mathrm{d}x\,\mathrm{d}y$$

so that, performing the integration by parts on left-hand side of equation (7), we obtain

$$\frac{\mu}{2}[x\phi^2(x)]_{-a/2}^{a/2} - \frac{\mu}{2}\int_{-a/2}^{a/2} \phi^2(x)\,\mathrm{d}x = \frac{1}{2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} (x-y)\phi(x)\phi(y)\frac{\mathrm{d}}{\mathrm{d}x}K(x-y)\,\mathrm{d}x\,\mathrm{d}y. \tag{8}$$

Multiply both members in equation (4) by $\phi(x)$ and integrate in $(-a/2, a/2)$. If the resulting expression is inserted in left-hand side of equation (8), we obtain

$$\mu[x\phi^2(x)]_{-a/2}^{a/2} = -\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} \phi(x)\phi(y)\left(K(x-y) + (x-y)\frac{\mathrm{d}}{\mathrm{d}x}K(x-y)\right)\mathrm{d}x\,\mathrm{d}y. \tag{9}$$

The kernel in large parentheses in equation (9), which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}[(x-y)K(x-y)]$$

is easily seen to coincide with $-T(x-y)$ under the hypotheses previously given for $p(\nu)$, so that equation (9) becomes

$$\tfrac{1}{2}\mu a(\phi^2(\tfrac{1}{2}a) + \phi^2(-\tfrac{1}{2}a)) = \int_{-a/2}^{a/2}\int_{-a/2}^{a/2} \phi(x)\phi(y)T(x-y)\,\mathrm{d}x\,\mathrm{d}y. \tag{10}$$

It follows from equation (10) that $\phi(x)$ cannot vanish simultaneously in $\pm\frac{1}{2}a$, when $T(x)$ is definite; thus the lemma is proved.

Observe that, exploiting equation (6), the following relation holds:

$$\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} \phi(x)\phi(y)T(x-y)\,\mathrm{d}x\,\mathrm{d}y = \int_{-\nu_M}^{\nu_M} \nu p'(\nu)|\tilde{\phi}_{\mathrm{T}}(\nu)|^2\,\mathrm{d}\nu \tag{11}$$

where $\tilde{\phi}_{\mathrm{T}}(\nu)$ is the FT of $\phi(x)$ truncated to the interval $(-a/2, a/2)$. Relation (11) is useful to recognize whether $T(x)$ is definite. Indeed $\phi_{\mathrm{T}}(\nu)$ is an analytic function and cannot vanish everywhere in the interval $(-\nu_M, \nu_M)$ so that it will suffice that $\nu p'(\nu)$ has a constant sign in order to have $T(x)$ definite. In particular, it is easily seen that $T(x)$ is definite when $p(\nu)$ equals $\Lambda(\nu)$, ie $K(x)$ equals $\mathrm{sinc}^2(x)$. Using this lemma, non-degeneracy is easily obtained. Let us first observe that if an eigenfunction $\phi(x)$ is to be neither even nor odd, both its even and odd parts, defined as

$$\phi_{\mathrm{e}}(x) = \tfrac{1}{2}(\phi(x)+\phi(-x)) \qquad \phi_{\mathrm{o}}(x) = \tfrac{1}{2}(\phi(x)-\phi(-x))$$

satisfy integral equation (4), with the same eigenvalue. By the same argument as used by Slepian and Pollack (1961) with reference to the sinc kernel, if two degenerate eigenfunctions, an even one and an odd one, corresponding to the same eigenvalue exist, the following relation holds:

$$\phi_{\mathrm{e}}(a/2)\phi_{\mathrm{o}}(a/2) = 0 \tag{12}$$

so that one of them should vanish both at $\pm\frac{1}{2}a$.

From this fact and from the lemma it follows that two degenerate eigenfunctions of this kind cannot exist; the preceding discussion also excludes existence of eigenfunctions neither even nor odd. Finally, if there should exist two degenerate eigenfunctions both even (or odd), we could always find a linear combination of them with both coefficients different from zero, which is an eigenfunction and vanishes at $\pm\frac{1}{2}a$. But from the lemma this is impossible, and thus we can conclude that there must be non-degeneracy of eigenvalues for integral equation (4).

3. Behaviour of the eigenvalues

3.1. General remarks

It is a known feature of equation (3) that its eigenvalues tend to decrease almost linearly with respect to the order index (Fedotowski 1972, Gori 1974, Bendinelli *et al* 1974) and they become exceedingly small beyond a critical index equal to $[2a]$. A rough approximation of this behaviour is given by the formula (Bendinelli *et al* 1974):

$$\begin{aligned} \mu_n &= 1-\frac{n}{2a} \qquad && n \lesssim 2a \\ \mu_n &= 0 && n \gtrsim 2a. \end{aligned} \tag{13}$$

This expression does not give the correct value of the eigenvalue sum, being

$$\sum_{n=0}^{\infty} \mu_n = a+\tfrac{1}{2}$$

whereas the eigenvalue sum should equal the space-bandwidth product a (Gori 1974). Another approximate formula has been given by Fedotowski (1972)

$$\begin{aligned} \mu_n &= \frac{[2a]-n+\delta}{[2a]+1} \qquad n \leqslant [2a] \\ \mu_n &= 0 \qquad n > [2a] \end{aligned} \tag{14}$$

where $\delta = a-\frac{1}{2}[2a]$. This formula gives the correct result for the eigenvalue sum.

Among eigenfunctions and eigenvalues, $\phi_0(x)$ and μ_0 have a special importance, because $\phi_0(x)$ is the only eigenfunction with constant sign within the interval $(-a/2, a/2)$; this makes $\phi_0(x)$ of particular physical meaning (Bendinelli *et al* 1974). As a consequence it is important to inquire about the dependence of μ_0 on the space-bandwidth product a. We observe that equation (13) gives a unity value for μ_0 regardless of a. According to equation (14), μ_0 increases from 0 to 1, when a ranges from 0 to ∞. Nevertheless the increase law is oscillatory, whereas a monotone behaviour should be expected. In the following section we will establish an upper bound for μ_0, which exhibits a monotone dependence on a and suggests another approximate law for the eigenvalue behaviour.

3.2. *Upper limitation for μ_0*

For any function f defined in the space of functions square summable in the interval $(-a/2, a/2)$ we can build the Rayleigh ratio

$$\frac{\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} \operatorname{sinc}^2(x-y)f(x)f^*(y)\,dx\,dy}{\int_{-a/2}^{a/2} f(x)f^*(x)\,dx} = \frac{1}{E}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} \operatorname{sinc}^2(x-y)f(x)f^*(y)\,dx\,dy$$

where $E = \int_{-a/2}^{a/2} |f(x)|^2\,dx$. The first eigenvalue μ_0 equals the maximum reached by the Rayleigh ratio (Riesz and Nagy 1955):

$$\mu_0 = \max\frac{1}{E}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} \operatorname{sinc}^2(x-y)f(x)f^*(y)\,dx\,dy = \max\frac{1}{E}\int_{-1}^{1} \Lambda(\nu)|\tilde{f}(\nu)|^2\,d\nu \tag{15}$$

where the sinc² function has been expressed through its FT $\Lambda(\nu)$.

The function $f(x)$ is different from zero only in the interval $(-a/2, a/2)$. Therefore, its FT $\tilde{f}(\nu)$, which has $f(x)$ as 'spectrum', is a band-limited function. As a consequence, $\tilde{f}(\nu)$ must satisfy the inequality (Papoulis 1968)

$$|\tilde{f}(\nu)|^2 \leqslant aE. \tag{16}$$

We now look for the function $\tilde{f}(\nu)$, which maximizes the Rayleigh ratio (15). Due to the shape of $\Lambda(\nu)$ the function $\tilde{f}(\nu)$ has to be mostly concentrated near the origin. On the other hand, $|\tilde{f}(\nu)|^2$ cannot be greater than aE. Hence, the best shape for $|\tilde{f}(\nu)|^2$ would be approximately rectangular, centred on the origin, of height aE and limited within the interval $(-1/2a, 1/2a)$. The last limitation follows from the Parseval theorem

$$\int_{-\infty}^{\infty} |\tilde{f}(\nu)|^2\,d\nu = E.$$

Although this behaviour cannot hold, strictly speaking, for a band-limited function ($\tilde{f}(\nu)$ is an analytic function and cannot vanish everywhere outside $(-1/2a, 1/2a)$), it surely gives an upper bound for the concentration of $|\tilde{f}(\nu)|^2$. Therefore, to get an upper limit for μ_0 we have simply to calculate the Rayleigh ratio corresponding to this situation. Two cases are possible according to the value of a, as shown in figure 1.

Computing μ_0 in the two situations we have:

$$\begin{aligned}\mu_0 &\leqslant \frac{1}{E}\int_{-1/2a}^{1/2a} \Lambda(\nu)aE\,\mathrm{d}\nu = 1-\frac{1}{4a} \qquad (a \geqslant \tfrac{1}{2})\\ \mu_0 &\leqslant \frac{1}{E}\int_{-1}^{1} \Lambda(\nu)aE\,\mathrm{d}\nu = a \qquad (a < \tfrac{1}{2}).\end{aligned} \tag{17}$$

Equation (17) represents the limitation we were looking for.

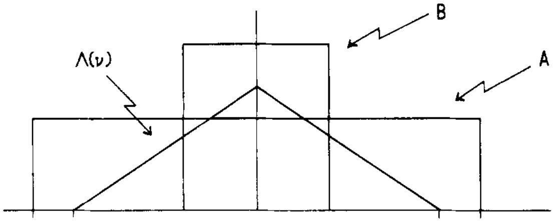

Figure 1. Best shape of $|\tilde{f}(\nu)|^2$ superimposed on $\Lambda(\nu)$ for: A, space-bandwidth product $a < \frac{1}{2}$; B, $a > \frac{1}{2}$.

3.3. Behaviour of the eigenvalues

The existence of the upper bound for μ_0 expressed by equation (17) suggests that, for $a > \frac{1}{2}$, the eigenvalues can be approximated by the formula

$$\begin{aligned}\mu_n &\simeq 1-\frac{1}{4a}-\frac{n}{2a} \qquad n \lesssim [2a-\tfrac{1}{2}]\\ \mu_n &\simeq 0 \qquad n \gtrsim [2a-\tfrac{1}{2}].\end{aligned} \tag{18}$$

Equation (18) incorporates the upper bound for μ_0 (when $a > \frac{1}{2}$). Furthermore, it satisfies the condition

$$\sum_{n=0}^{\infty} \mu_n = a.$$

Comparison of values of μ_n computed numerically with values obtained from formula (18) shows also that this formula gives a better approximation than formula (13), without having the oscillatory feature of equation (14). This approximation, for a greater than some units, is better than 2%, except for the last eigenvalue, ie the smallest, corresponding to $n = [2a-\frac{1}{2}]$.

4. Application to photocounting and speckle statistics

We now apply the preceding results to the classical analysis of Kac and Siegert (1947) on square-law detection of noise. Two different transcriptions of this analysis exist in the realm of optics. The first deals with the time intensity fluctuations of a Gaussian light field (Mehta 1971) whereas the second refers to speckle statistics (Barakat 1973, Dainty 1971). Here we adopt the language of speckle statistics but, of course, our results apply also to the first problem.

Suppose a spatially coherent beam of quasi-monochromatic light is incident on a static diffuser (ground glass). We refer to the speckle intensity distribution $I(x)$ in the far-field in the one-dimensional case. Define the integrated intensity

$$W=\int_{-a/2}^{a/2} I(x)\,\mathrm{d}x \tag{18}$$

to take into account the finite aperture $(-a/2, a/2)$ of the detector. Under the usual assumptions (Barakat 1973) the probability density of $I(x)$ is negative exponential, whereas the probability density for W depends on the field correlation function in the detector plane. The Kac–Siegert analysis leads to the following expression for the characteristic function $\tilde{P}_W(v)$ of $P_W(W)$:

$$\tilde{P}_W(v)=\prod_n \frac{1}{1-2\pi i n\gamma_n} \tag{19}$$

where the γ_n are the eigenvalues of the integral equation

$$\gamma_n\phi_n(x)=\int_{-a/2}^{a/2}\phi_n(y)R(x,y)\,\mathrm{d}y. \tag{20}$$

Here, $R(x, y)$ is the correlation function for the field at the two points x and y in the speckle pattern. Fourier transformation of expression (19) leads to the probability density function $P_W(W)$. If there is no degeneracy of the eigenvalues γ_n the result is

$$P_W(W)=\sum_{n=0}^{\infty}\frac{C_n}{\gamma_n}\mathrm{e}^{-W/\gamma_n} \tag{21}$$

with

$$C_n=\prod_s{}'\left(1-\frac{\gamma_s}{\gamma_n}\right)^{-1} \tag{22}$$

where Π' means that $s = n$ is excluded. Equations (21) and (22) give a mean value $\overline{W}=\Sigma_n\gamma_n$.

If some of the eigenvalues are degenerate, more cumbersome formulae are required (Barakat 1973). The limiting case is that in which only the first eigenvalue is appreciably different from zero and has a degeneracy of order N, all the other eigenvalues being exceedingly small. This case leads to the Γ distribution (Scribot 1974).

Let us refer to the case of a sinc² correlation function

$$R_1(x,y)=\mathrm{sinc}^2(x-y) \tag{23}$$

where a constant proportionality factor depending on field intensity has been omitted. This kind of correlation function would be obtained in the far-field of a diffuser with an intensity transmission function of triangular shape.

It has been shown in § 2 that the eigenvalues of equation (3) are not degenerate. Therefore, we can use equations (21) and (22) to compute the probability density for the integrated intensity. As a reference case, we will also consider a correlation function

$$R_2(x,y)=\mathrm{sinc}(x-y) \tag{24}$$

that would correspond to a diffuser with an intensity transmission function of rectangular shape. In this case the eigenvalues are those of the prolate spheroidal wavefunctions. They are also non-degenerate (Slepian and Pollack 1961). The total area of the FT of both functions (23) and (24) is unity so that the space-bandwidth product a is the same for the two cases we are considering.

With reference to the case of the R_1 kernel, we first compare the results obtained through equations (21) and (22) when the exact eigenvalues of the sinc^2 kernel are used and when the approximate formula (18) is used. Some curves are shown in figures 2 and 3. They refer to the probability density of $W/\bar{W}$ for $a = 3{\cdot}5$ (figure 2) and $a = 7$ (figure 3). Curves E give the exact results obtained by use of eigenvalues determined numerically by Bendinelli *et al* (1974), whereas curves A refer to the approximate results obtained by use of formula (18) for the eigenvalues. The differences between the exact and the approximate curves do not exceed a few per cent. As it was to be expected differences are lower for $a = 7$ than for $a = 3{\cdot}5$. Without going into a detailed analysis

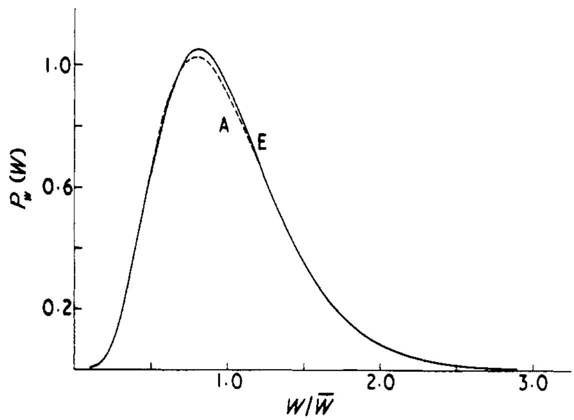

Figure 2. Probability density function for $a = 3{\cdot}5$. Curve E is obtained with exact results for eigenvalues μ_n; curve A is obtained with approximate values from formula (18).

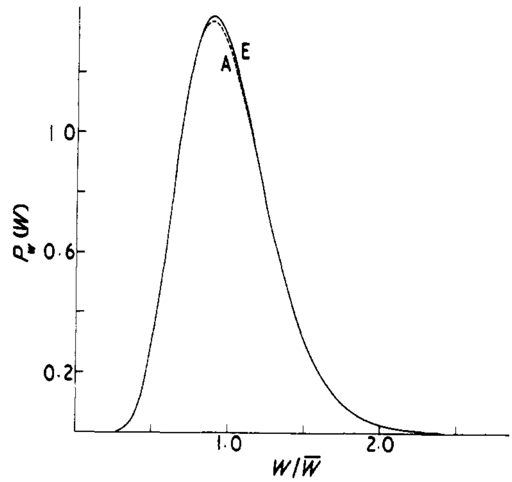

Figure 3. Same as figure 2 for the case $a = 7$.

of the behaviour of these differences with respect to the value of a, we conclude that when a is of the order of some units or greater, the approximate formula can be used with an accuracy of a few per cent.

Let us now compare the results with R_1 and R_2 kernels (see expressions (23) and 24)). An example is shown in figure 4. Curve T gives the probability density for $W/\overline{W}$ when the diffuser has a triangularly shaped intensity transmission function (sinc^2 correlation). The space-bandwidth product is $a = 5{\cdot}09$ and the eigenvalues are computed through

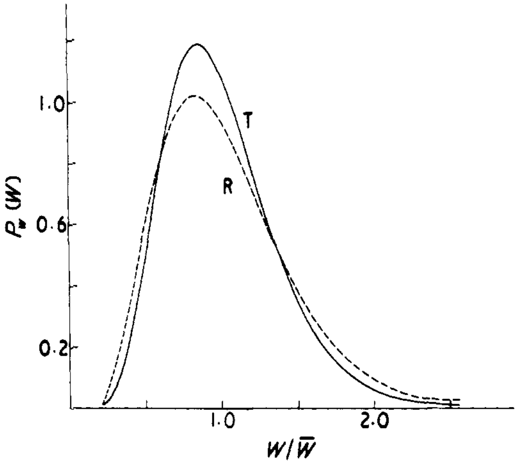

Figure 4. Probability density function for $a = 5{\cdot}09$. Curve T for sinc^2 correlation eigenvalues; curve R for sinc correlation eigenvalues.

the approximate formula (18), the exact eigenvalue being not available. Curve R refers to a diffuser with a rectangularly shaped intensity transmission function (sinc correlation). The space-bandwidth product is again $a = 5{\cdot}09$ and the eigenvalues are those given by Slepian and Sonnenblick (1965). We see that curve T is more peaked near the mean value $W/\overline{W} = 1$ than curve R. The local difference between the two curves reaches 15% and, therefore, cannot be ascribed to the use of the approximate formula (18) for the eigenvalues of sinc^2 kernel. In figure 5 curve T (sinc^2 kernel) for $a = 3$ is compared with two different curves R′ and R″ (sinc kernel) corresponding to $a = 3{\cdot}8$ and $a = 4{\cdot}5$, respectively. Curve T is situated between the two curves R′ and R″ so that, roughly speaking, curve T for $a = 3$ would match with a curve R corresponding to $a \simeq 4$. From figures 4 and 5 we conclude that the probability density function we obtain for a given space-bandwidth product and the sinc^2 kernel is similar to that we would obtain with the sinc kernel, provided the space-bandwidth product is suitably increased. A qualitative explanation of this result is as follows. The Kac–Siegert analysis is equivalent to representing the integrated intensity as a sum of independent random variables. Each variable has a negative exponential probability density with a mean value γ_n. For both the sinc and sinc^2 kernels the eigenvalues become exceedingly small when their order index n is greater than a certain critical index (Slepian and Pollack 1961, Gori 1974). Such a critical index equals $[a]$ for the sinc kernel, whereas it equals $[2a]$ for the sinc^2 kernel. Therefore, discarding eigenvalues beyong the critical index, the integrated intensity W is given by the sum of $[a]$ or $[2a]$ independent variables for the sinc and sinc^2 kernels, respectively. It is a well known feature of equation (21) that by increasing

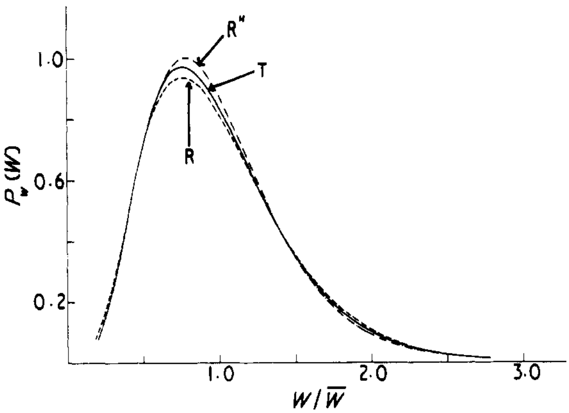

Figure 5. Comparison between probability density functions obtained with sinc^2 and sinc correlation eigenvalues. Curve R′ is for sinc correlation eigenvalues for the case $a = 3{\cdot}8$. Curve R″ is the same as R′, for the case $a = 4{\cdot}5$; curve T for sinc^2 correlation eigenvalues in the case $a = 3$.

the number of non-negligible terms of the sum, the probability density for W tends to narrow near $(W/\overline{W}) = 1$. This explains why in figure 4 curve T is more peaked near $(W/\overline{W}) = 1$ than curve R. Analogous considerations can be applied to figure 5. In this elementary explanation we did not take into account that the eigenvalues of the sinc^2 kerenel decrease almost linearly with their order index before the critical index $[2a]$ is reached, whereas the eigenvalues of the sinc kernel are nearly equal to each other for indices smaller than the critical index $[a]$. As a matter of fact, the increase from $[a]$ to $[2a]$ non-negligible eigenvalues seems to be the main factor that affects the probability density function. The dependence of the probability densities for integrated intensity on the shape of the kernel R has been subjected to previous investigations (Barakat 1973, Mehta and Mehta 1973) with reference to sinc, Gaussian and negative exponential kernels.

Further work is required to coordinate all these results and to proceed to a critical investigation which could arrive at general conclusions.

This is beyond the aim of this paper; we do not dwell on this matter here.

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